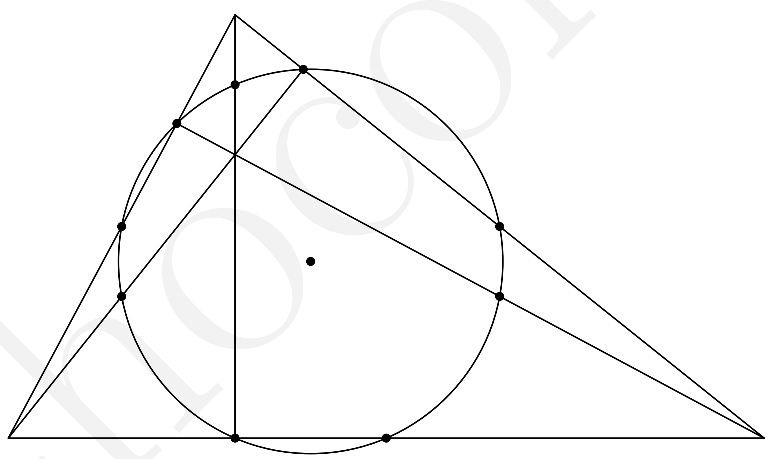
幾何秘笈

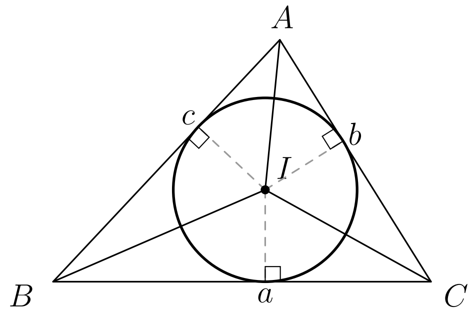
Chocomint



Contents

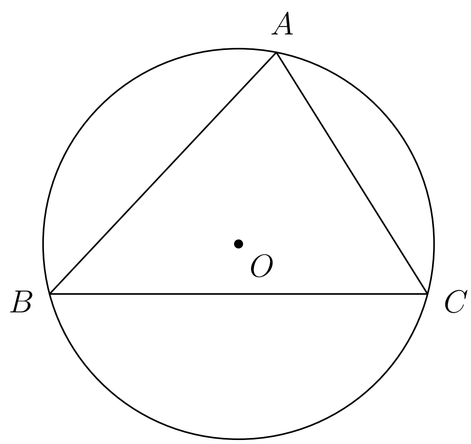
1 三角形的三心與垂心

1.1 內心 (Incenter)

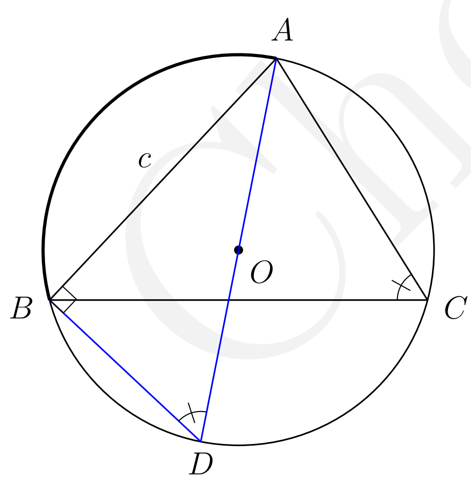


1. 內接圓圓心
2. 到三邊等距
3. 三頂點角平分線
4. $r = \dfrac{\text{面積}}{\text{半周長}} = \dfrac{\triangle}{s}$

1.2 外心 (Circumcenter)



1. 外接圓圓心
2. 到三頂點等距
3. 三垂直平分線交點
4. 直角三角形的外心在斜邊中點
5. $R = \dfrac{abc}{4\triangle}$



Proof.

做$\overrightarrow{AO}$，交圓O於D

$\angle ACB = \angle ADB$ (對同弧)

$\overline{AD} = 2R$

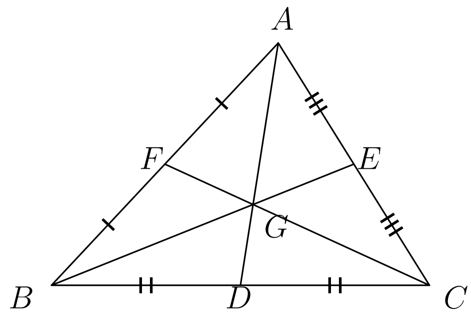
$\therefore \sin C = \dfrac{c}{2R}$

$\triangle = \dfrac{1}{2}ab\sin C \Rightarrow \sin C = \dfrac{2\triangle}{ab}$

$\Rightarrow \dfrac{c}{2R} = \dfrac{2\triangle}{ab} \Rightarrow R = \dfrac{abc}{4\triangle}$

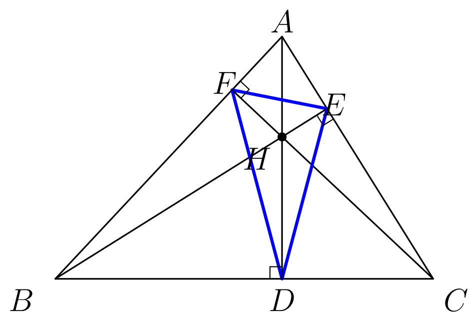
□

1.3 重心 (Center of Gravity)



1. 中線交點
2. 六塊面積相等
3. $\dfrac{\overline{GD}}{\overline{AG}} = \dfrac{\overline{GE}}{\overline{BG}} = \dfrac{\overline{GF}}{\overline{CG}} = \dfrac{1}{2}$
4. $G = \dfrac{1}{3}(A + B + C)$

1.4 垂心 (Orthocenter)

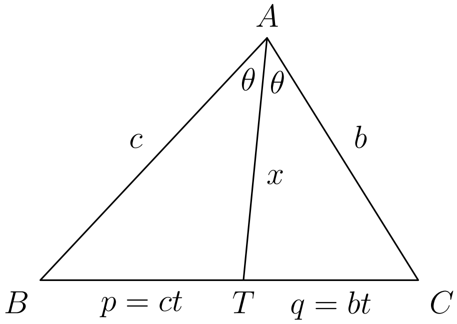


1. 三垂線交點
2. 「垂足三角形」($\triangle DEF$)爲內部三角形中周長最小的三角形，且 H 爲垂足三角形之內心

2 分線定理

2.1 角平分線定理

定理 2.1 (Schooten's Theorem). $\overline{AT}^2 = \overline{AB} \times \overline{AC} - \overline{BT} \times \overline{CT}$



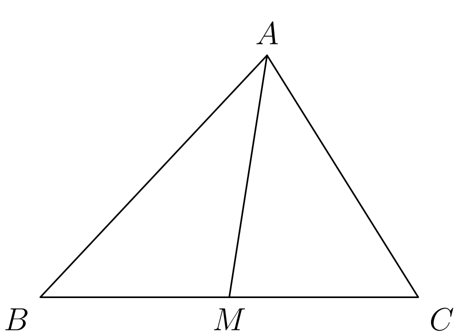
Proof. 由面積相等列式：

$$
\begin{aligned}
&\frac{1}{2} bc \sin 2\theta = \frac{1}{2} bx \sin\theta + \frac{1}{2} cx \sin\theta \\
&\Rightarrow 2bc\cos\theta = x(b+c) \\
&\Rightarrow x = \frac{2bc}{b+c}\cos\theta = \frac{2bc}{b+c}\cos\frac{A}{2} \\
&\quad = \frac{2bc}{b+c} \times \sqrt{\frac{1+\cos A}{2}} \\
&\quad = \frac{2bc}{b+c} \times \sqrt{\frac{1+\frac{b^2+c^2-a^2}{2bc}}{2}} \\
&\quad = \frac{2bc}{b+c} \times \sqrt{\frac{(b+c)^2-a^2}{4bc}} \\
&\quad = \sqrt{\frac{bc\left[(b+c)^2-a^2\right]}{(b+c)^2}} \\
&\quad = \sqrt{bc\left[1-\left(\frac{a}{b+c}\right)^2\right]} \\
&\quad = \sqrt{bc-\left(\frac{ab}{b+c}\right)\left(\frac{ac}{b+c}\right)} \\
&\quad = \sqrt{bc-pq}
\end{aligned}
$$

□

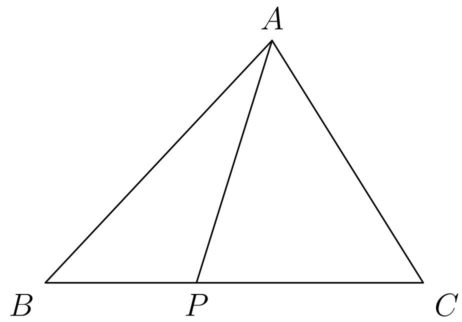
2.2 中線定理

定理 2.2 (Apollonius's Theorem). $\overline{AB}^2 + \overline{AC}^2 = 2\left(\overline{AM}^2 + \overline{BM}^2\right)$



2.3 斯圖爾特定理

定理 2.3 (Stewart's Theorem). $\overline{PC}\,\overline{AB}^2+\overline{PB}\,\overline{AC}^2=\overline{BC}\left(\overline{PA}^2+\overline{PB}\,\overline{PC}\right)$

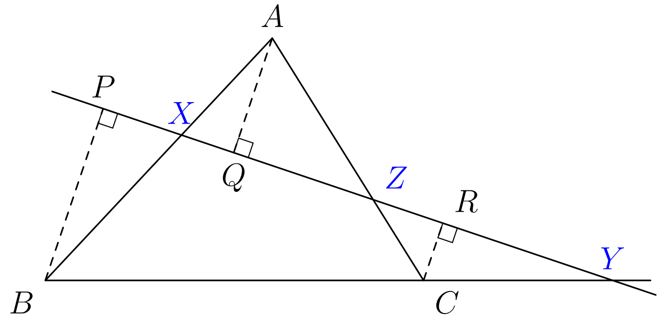


3 圓的性質

4 三角形的共線點定理

4.1 孟氏定理

定理 4.1 (Menelaus' Theorem). $\dfrac{\overline{AX}}{\overline{XB}}\cdot\dfrac{\overline{BY}}{\overline{YC}}\cdot\dfrac{\overline{CZ}}{\overline{ZA}}=1\iff XYZ$ 三點共線



Proof. ($\Leftarrow$) 做垂線 $\overline{AQ}$, $\overline{BP}$, $\overline{RC}$

$$\overline{AX}:\overline{BX}=\overline{AQ}:\overline{BP}$$
$$\overline{CZ}:\overline{ZA}=\overline{CR}:\overline{AQ}$$
$$\overline{BY}:\overline{YC}=\overline{BP}:\overline{CR}$$
$$\frac{\overline{AX}}{\overline{BX}}\cdot\frac{\overline{BY}}{\overline{YC}}\cdot\frac{\overline{CZ}}{\overline{ZA}}=\frac{\overline{AQ}}{\overline{BP}}\cdot\frac{\overline{BP}}{\overline{CR}}\cdot\frac{\overline{CR}}{\overline{AQ}}=1$$

($\Rightarrow$) 假設 XZ 交 BC 異於 Y 之點 Y'

由($\Leftarrow$)得知：$\dfrac{\overline{AX}}{\overline{XB}}\cdot\dfrac{\overline{BY'}}{\overline{Y'C}}\cdot\dfrac{\overline{CZ}}{\overline{ZA}}=1$

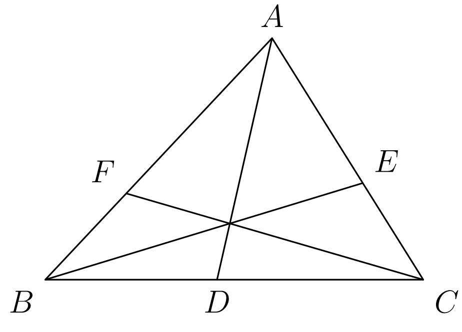
由題目條件得知：$\dfrac{\overline{AX}}{\overline{XB}}\cdot\dfrac{\overline{BY}}{\overline{YC}}\cdot\dfrac{\overline{CZ}}{\overline{ZA}}=1$

兩式比較可知，$\dfrac{\overline{BY'}}{\overline{Y'C}}=\dfrac{\overline{BY}}{\overline{YC}}\Rightarrow Y=Y'$

故 XYZ 三點共線 □

4.2 帥氏定理

定理 4.2 (Ceva's Theorem). $\dfrac{\overline{BD}}{\overline{DC}}\cdot\dfrac{\overline{CE}}{\overline{EA}}\cdot\dfrac{\overline{AF}}{\overline{FB}}=1\iff\overline{AD},\overline{BD},\overline{CF}$ 三線共點



5 海龍公式

定理 5.1 (Heron's Formula)**.**

$$\triangle = \sqrt{s(s-a)(s-b)(s-c)},\ s = \frac{a+b+c}{2}$$

Proof.

$$\begin{aligned}
\triangle &= \frac{1}{2}bc\sin A \\
\triangle^2 &= \frac{1}{4}b^2c^2\sin^2 A \\
&= \frac{1}{4}b^2c^2(1-\cos^2 A) \\
&= \frac{1}{4}b^2c^2(1+\cos A)(1-\cos A) \\
&= \frac{1}{4}b^2c^2\left(1+\frac{b^2+c^2-a^2}{2bc}\right)\left(1-\frac{b^2+c^2-a^2}{2bc}\right) \\
&= \frac{1}{4}b^2c^2\left(\frac{b^2+c^2-a^2+2bc}{2bc}\right)\left(\frac{a^2-b^2-c^2+2bc}{2bc}\right) \\
&= \frac{1}{16}\left((b+c)^2-a^2\right)\left(a^2-(b-c)^2\right) \\
&= \frac{1}{16}(b+c+a)(b+c-a)(a+b-c)(a-b+c) \\
&= \left(\frac{b+c+a}{2}\right)\left(\frac{b+c-a}{2}\right)\left(\frac{a+b-c}{2}\right)\left(\frac{a-b+c}{2}\right) \\
&= s(s-a)(s-b)(s-c) \\
\text{這裡令 } s &= \frac{a+b+c}{2} \Rightarrow \triangle = \sqrt{s(s-a)(s-b)(s-c)}
\end{aligned}$$

□

定理 5.2 (Brahmagupta's Formula)**.** 對於任意圓内接四邊形而言，面積由下式給出：

$$\triangle = \sqrt{(s-a)(s-b)(s-c)(s-d)},\ s = \frac{a+b+c+d}{2}$$

Proof. [略]

6 三角函數

6.1 正弦定理

定理 6.1 (Law of Sines). 對於任意三角形，有以下關係：

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

6.2 餘弦定理

定理 6.2 (Law of Cosines). 對於任意三角形，有以下關係：

$$c^2 = a^2 + b^2 - 2ab\cos C$$

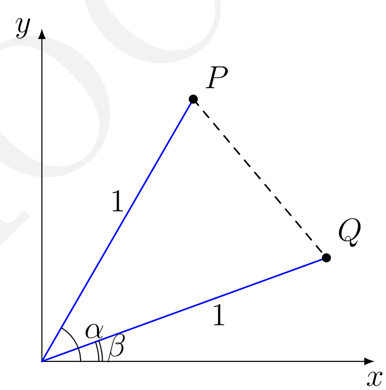
或者更常使用這個形式：

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

6.3 和角公式

$$\begin{aligned}
\sin(\alpha + \beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\
\sin(\alpha - \beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\
\cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
\cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta
\end{aligned}$$

Proof. 由 cos 的差角開始



$$\begin{aligned}
\overline{PQ} &= \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2} \\
&= \sqrt{\cos^2\alpha + \cos^2\beta + \sin^2\alpha + \sin^2\beta - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta} \\
\overline{PQ}^2 &= 1 + 1 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \\
&\overset{LC}{=} 1^2 + 1^2 - 2\times 1\times 1\times\cos(\alpha - \beta) \\
\therefore \cos(\alpha - \beta) &= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)
\end{aligned}$$

□

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$$
$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$$

6.4 倍角公式

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$
$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$
$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

6.5 半角公式

7 莫雷三分角定理

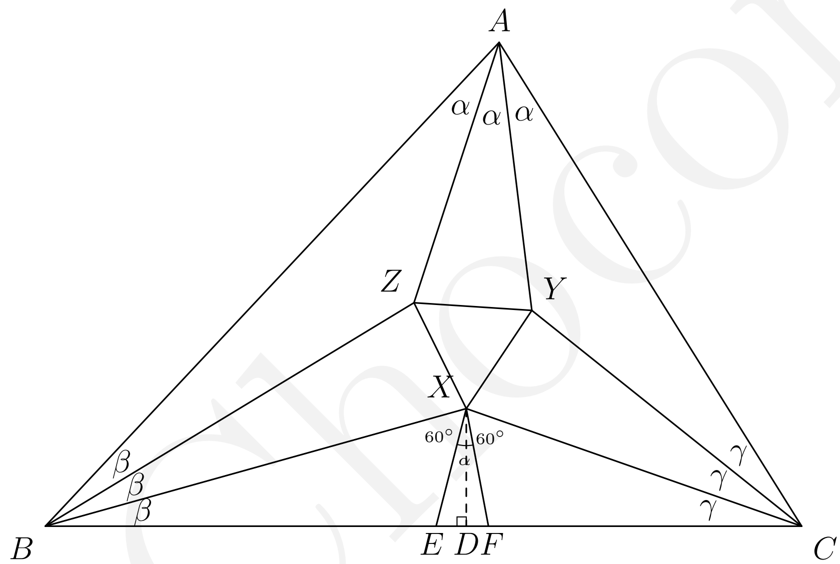
引理 7.1. $\sin 3\theta = 4\sin\theta\sin(60^\circ+\theta)\sin(120^\circ+\theta)$

Proof.

$$\begin{aligned}\sin 3\theta &= 3\sin\theta - 4\sin^3\theta\\ &= \sin\theta\left(3-4\sin^2\theta\right)\\ &= \sin\theta\left(3\cos^2\theta-\sin^2\theta\right)\\ &= \sin\theta\left(\sqrt{3}\cos\theta+\sin\theta\right)\left(\sqrt{3}\cos\theta-\sin\theta\right)\\ &= 4\sin\theta\left(\frac{\sqrt{3}}{2}\cos\theta+\frac{1}{2}\sin\theta\right)\left(\frac{\sqrt{3}}{2}\cos\theta-\frac{1}{2}\sin\theta\right)\\ &= 4\sin\theta\left(\sin 60^\circ\cos\theta+\cos 60^\circ\sin\theta\right)\left(\sin 120^\circ\cos\theta-\cos 120^\circ\sin\theta\right)\\ &= 4\sin\theta\sin(60^\circ+\theta)\sin(120^\circ+\theta)\end{aligned}$$

□

定理 7.2 (Morley's Theorem)**.** 對於任意的三角形，其三個內角作角三分線，靠近公共邊三分線的三個交點，是一個正三角形。



Proof.

1. 作 E 於 $\overline{BC}$ 上使得 $\angle BXE = 60^\circ$
2. 作 F 於 $\overline{BC}$ 上使得 $\angle CXF = 60^\circ$

$$\begin{aligned}&\because 3\alpha+3\beta+3\gamma=180^\circ\\ &\Rightarrow \alpha+\beta+\gamma=60^\circ\\ &\Rightarrow \angle EXF=\alpha\\ &\Rightarrow \angle BXC=120^\circ+\alpha\end{aligned}$$

同理，$\begin{cases}\angle CYA = 120^\circ + \beta \\ \angle AZB = 120^\circ + \gamma\end{cases}$

$$\begin{cases}\sin(120^\circ + \beta) = \dfrac{\overline{AC}\sin\gamma}{\overline{AY}} \\ \sin(120^\circ + \gamma) = \dfrac{\overline{AB}\sin\beta}{\overline{AZ}}\end{cases}$$

$$\begin{cases}\angle XEF = 60^\circ + \beta \Rightarrow \sin(60^\circ + \beta) = \dfrac{\overline{XD}}{\overline{XE}} \\ \angle XFE = 60^\circ + \gamma \Rightarrow \sin(60^\circ + \gamma) = \dfrac{\overline{XD}}{\overline{XF}}\end{cases}$$

$$\begin{aligned}\frac{\overline{AC}}{\sin 3\beta} &= \frac{\overline{AB}}{\sin 3\gamma} \\ \Rightarrow \overline{AB}\sin 3\beta &= \overline{AC}\sin 3\gamma \\ \Rightarrow \overline{AB}\sin\beta\sin(60^\circ + \beta)\sin(120^\circ + \beta) &= \overline{AC}\sin\gamma\sin(60^\circ + \gamma)\sin(120^\circ + \gamma) \\ \Rightarrow \overline{AB}\sin\beta\frac{\overline{XD}}{\overline{XE}}\frac{\overline{AC}\sin\gamma}{\overline{AY}} &= \overline{AC}\sin\gamma\frac{\overline{XD}}{\overline{XF}}\frac{\overline{AB}\sin\beta}{\overline{AZ}} \\ \Rightarrow \frac{\overline{AZ}}{\overline{XE}} &= \frac{\overline{AY}}{\overline{XF}} \\ \therefore \triangle AZY &\sim \triangle XEF\,(SAS)\end{aligned}$$

$\Rightarrow \begin{cases}\angle AZY = 60^\circ + \beta \\ \angle AZB = 180^\circ - \alpha - \beta \ \Rightarrow \angle XZY = 60^\circ \\ \angle BZX = 60^\circ + \alpha\end{cases}$

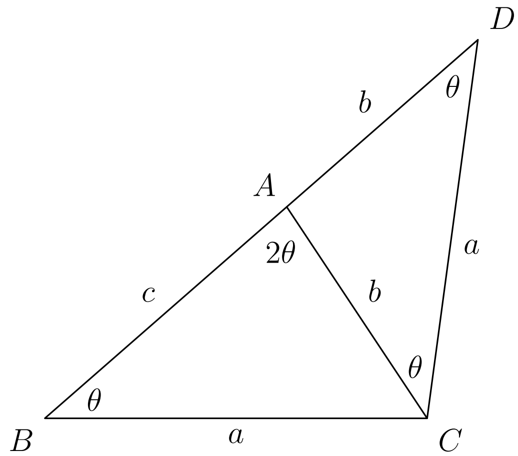
同理，$\angle ZXY = \angle XYZ = \angle XZY = 60^\circ$
故$\triangle XYZ$ 爲正三角形

□

8 整倍角關係

8.1 二倍角關係

定理 8.1. $a^2 = b(b+c)$, $\angle A = 2\angle B$



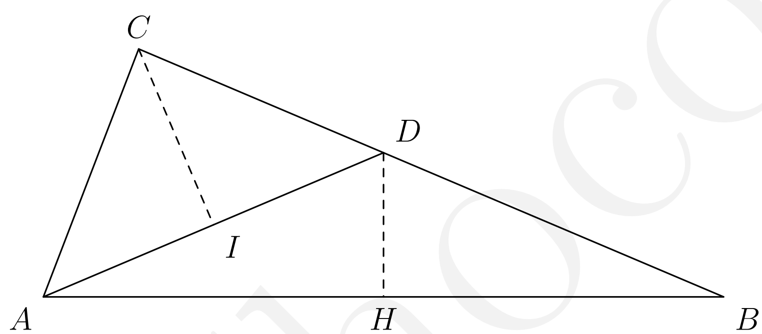
Proof. 延長$\overrightarrow{BA}$，使 D 於 $\overrightarrow{BA}$ 上，
且 $\angle ACD = \angle ABC = \theta$

$$\overline{AD} = b,\ \overline{CD} = a$$
$$\triangle ACD \sim \triangle CBD$$
$$\Rightarrow b : a = a : (b+c)$$
$$\Rightarrow a^2 = b(b+c)$$

□

8.2 三倍角關係

定理 8.2. $c^2 = \dfrac{1}{b}(a+b)(a-b)^2$, $\angle A = 3\angle B$



Proof. 做 D 於 $\overline{BC}$ 上使 $\angle DAB = \angle DBA$

$\because \angle CDA = \angle DAB + \angle DBA$

$\therefore \angle CAD = \angle CDA$

$\Rightarrow \overline{CD} = \overline{CA} = b$

$\Rightarrow \overline{DB} = a - b = \overline{DA}$

設 $\angle DAB = \theta$, $\angle CAD = 2\theta$

$$\cos\theta = \frac{\frac{c}{2}}{a-b} = \frac{c}{2(a-b)}$$
$$\cos 2\theta = 2\cos^2\theta - 1$$
$$= 2\left[\frac{c}{2(a-b)}\right]^2$$
$$= \frac{c^2}{2(a-b)^2} - 1$$
$$= \frac{c^2 - 2(a-b)^2}{2(a-b)^2}$$

In $\triangle CAI$:

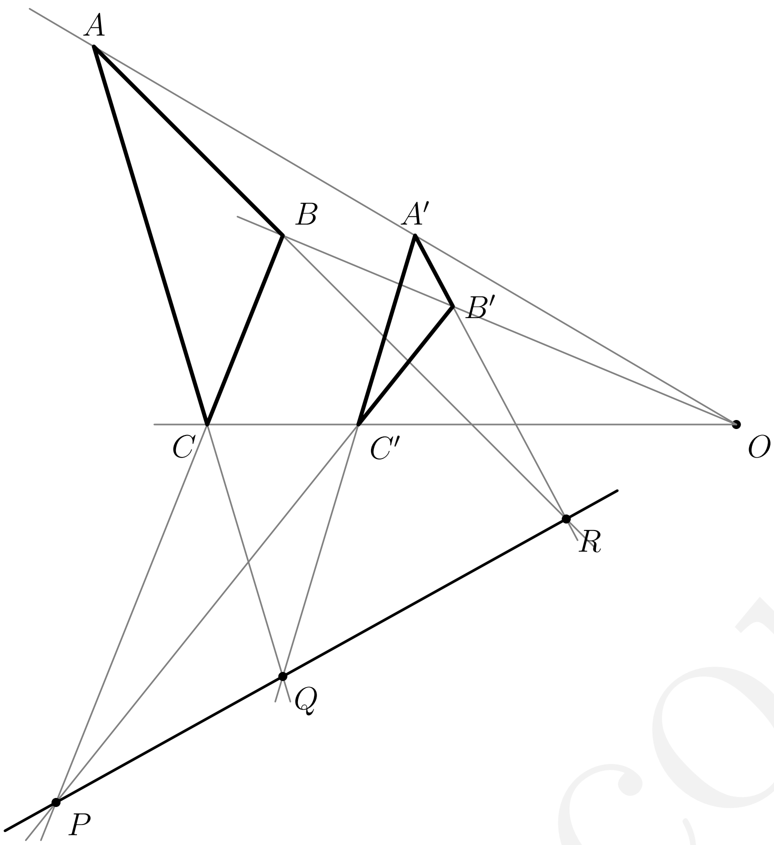
$$\cos 2\theta = \frac{\frac{a-b}{2}}{b}$$
$$= \frac{a-b}{2b}$$

$$\Rightarrow \frac{c^2 - 2(a-b)^2}{2(a-b)^2} = \frac{a-b}{2b}$$
$$\Rightarrow bc^2 - 2b(a-b)^2 = (a-b)^3$$
$$\Rightarrow bc^2 = (a-b)^2(a-b+2b)$$
$$\Rightarrow c^2 = \frac{1}{b}(a-b)^2(a+b)$$

□

9 笛沙格定理

定理 9.1 (Desargues' Theorem)**.** 任意兩個三角形對應頂點的連線共點若且唯若其對應邊的交點共線



Proof. B', C' 截過 $\triangle BCO$

$$\Rightarrow \frac{\overline{BP}}{\overline{PC}} \times \frac{\overline{CC'}}{\overline{C'O}} \times \frac{\overline{OB'}}{\overline{B'B}} = 1 \quad (1)$$

同理，

$$\begin{cases} \dfrac{\overline{CQ}}{\overline{QA}} \times \dfrac{\overline{AA'}}{\overline{A'O}} \times \dfrac{\overline{OC'}}{\overline{C'C}} = 1 \quad (2) \\ \dfrac{\overline{AR}}{\overline{RB}} \times \dfrac{\overline{BB'}}{\overline{B'O}} \times \dfrac{\overline{OA'}}{\overline{A'A}} = 1 \quad (3) \end{cases}$$

$(1) \times (2) \times (3)$

$$\Rightarrow \frac{\overline{BP}}{\overline{PC}} \times \frac{\overline{CC'}}{\overline{C'O}} \times \frac{\overline{OB'}}{\overline{B'B}} \times \frac{\overline{CQ}}{\overline{QA}} \times \frac{\overline{AA'}}{\overline{A'O}} \times \frac{\overline{OC'}}{\overline{C'C}} \times \frac{\overline{AR}}{\overline{RB}} \times \frac{\overline{BB'}}{\overline{B'O}} \times \frac{\overline{OA'}}{\overline{A'A}} = 1$$

$$\Rightarrow \frac{\overline{BP}}{\overline{PC}} \times \frac{\overline{CQ}}{\overline{QA}} \times \frac{\overline{AR}}{\overline{RB}} = 1$$

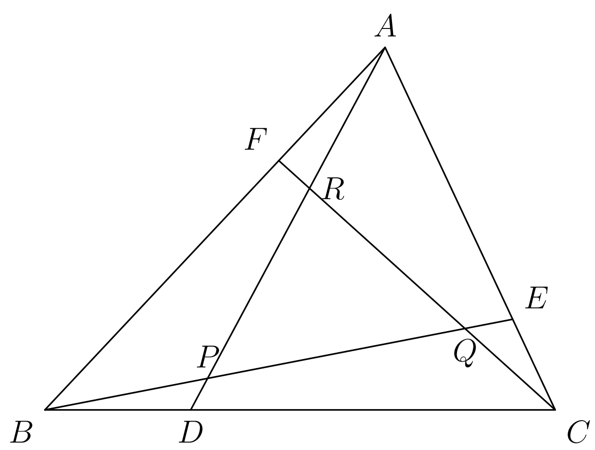
$\Rightarrow$ 由孟氏逆定理， ABC 爲三角形，故P, Q, R三點共線

□

10 羅斯定理

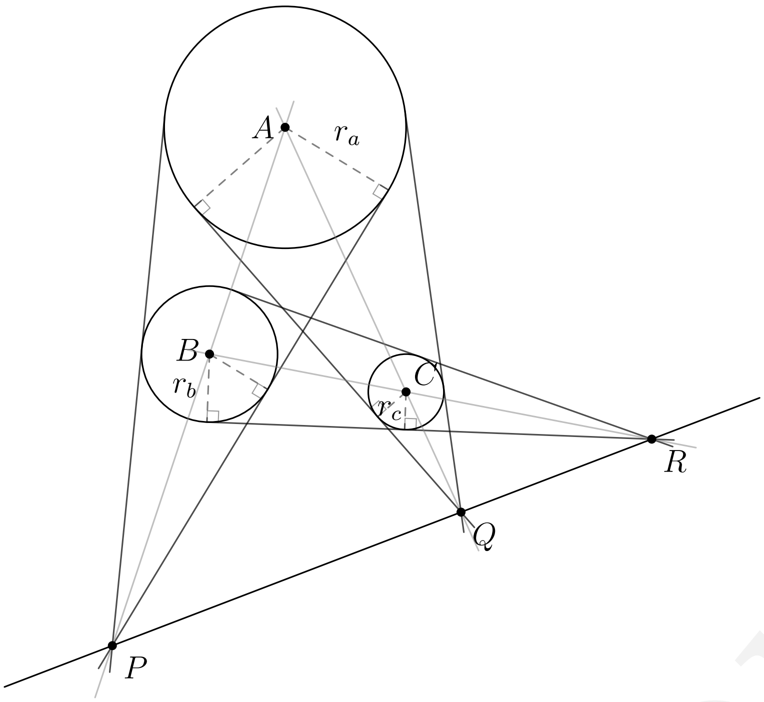
定理 10.1 (Routh's Theorem)**.** 若 $\overline{AF}:\overline{FB}=1:x$, $\overline{BD}:\overline{DC}=1:y$, $\overline{CE}:\overline{EA}=1:z$，則：

$$\frac{\triangle PQR}{\triangle ABC}=\frac{(xyz-1)^2}{(xz+x+1)(yx+y+1)(zy+z+1)}$$



11 蒙格定理

定理 11.1 (Monge's Theorem). 任三圓兩兩公切線交點共線



12 笛卡兒四圓定理

引理 12.1. 若 $\alpha+\beta+\gamma=2\pi$ ，則 $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1+2\cos\alpha\cos\beta\cos\gamma$

Proof. $\gamma=2\pi-(\alpha+\beta)\Rightarrow\cos\gamma=\cos(\alpha+\beta)$

$$\begin{aligned}
&\cos^2\alpha+\cos^2\beta+\cos^2\gamma\\
&=\cos^2\alpha+\cos^2\beta+\cos^2(\alpha+\beta)\\
&=\cos^2\alpha+\cos^2\beta+[\cos\alpha\cos\beta-\sin\alpha\sin\beta]^2\\
&=\cos^2\alpha+\cos^2\beta+\cos^2\alpha\cos^2\beta+\sin^2\alpha\sin^2\beta-2\sin\alpha\sin\beta\cos\alpha\cos\beta\\
&=\cos^2\alpha+\cos^2\beta+\cos^2\alpha\cos^2\beta+\left(1-\cos^2\alpha\right)\left(1-\cos^2\beta\right)-2\sin\alpha\sin\beta\cos\alpha\cos\beta\\
&=2\cos^2\alpha\cos^2\beta+1-2\sin\alpha\sin\beta\cos\alpha\cos\beta\\
&=2\left(\cos\alpha\cos\beta\right)^2-2\left(\sin\alpha\sin\beta\right)\left(\cos\alpha\cos\beta\right)+1\\
&=1+2\cos\alpha\cos\beta\left(\cos\alpha\cos\beta-\sin\alpha\sin\beta\right)\\
&=1+2\cos\alpha\cos\beta\cos(\alpha+\beta)\\
&=1+2\cos\alpha\cos\beta\cos\gamma
\end{aligned}$$

□

定理 12.2 (Descartes' Circle Theorem)**.** 若有三圓 A,B,C 相切，曲率各爲 k_1,k_2,k_3 ，且有一曲率爲 k_4 圓 O 與三圓相切，則：

$$2\left(k_1^2+k_2^2+k_3^2+k_4^2\right)=\left(k_1+k_2+k_3+k_4\right)^2$$

Proof. Let $\begin{cases}\alpha=\angle AOB\\ \beta=\angle BOC\\ \gamma=\angle COA\end{cases}$

$$\begin{aligned}
\cos\alpha&=\frac{(r_1+r_4)^2+(r_2+r_4)^2-(r_1+r_2)^2}{2(r_1+r_4)(r_2+r_4)}\\
&=\frac{2r_4^2+2r_4(r_1+r_2)-2r_1r_2}{2(r_1+r_4)(r_2+r_4)}\\
&=\frac{(r_1+r_4)(r_2+r_4)-2r_1r_2}{(r_1+r_4)(r_2+r_4)}\\
&=1-\frac{2r_1r_2}{(r_1+r_4)(r_2+r_4)}\\
&=1-\frac{2\left(\frac{1}{k_1}\right)\left(\frac{1}{k_2}\right)}{\left(\frac{1}{k_1}+\frac{1}{k_4}\right)\left(\frac{1}{k_2}+\frac{1}{k_4}\right)}\\
&=1-\frac{2k_4^2}{(k_1+k_4)(k_2+k_4)}\ :=1-\lambda_1
\end{aligned}$$

同理，

$$\cos\beta = 1 - \frac{2k_4^2}{(k_2+k_4)(k_3+k_4)} := 1-\lambda_2$$
$$\cos\gamma = 1 - \frac{2k_4^2}{(k_3+k_4)(k_1+k_4)} := 1-\lambda_3$$

將上式代入 **Lemma 12.1**：

$$\begin{aligned}
(1-\lambda_1)^2+(1-\lambda_2)^2+(1-\lambda_3)^2 &= 1+2(1-\lambda_1)(1-\lambda_2)(1-\lambda_3)\\
\lambda_1^2+\lambda_2^2+\lambda_3^2+2\lambda_1\lambda_2\lambda_3 &= 2(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)\\
\frac{\lambda_1}{\lambda_2\lambda_3}+\frac{\lambda_2}{\lambda_1\lambda_3}+\frac{\lambda_3}{\lambda_1\lambda_2}+2 &= 2\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}\right)\\
\frac{(k_4+k_1)^2}{2k_4^2}+\frac{(k_4+k_2)^2}{2k_4^2}+\frac{(k_4+k_3)^2}{2k_4^2}+2 &= 2\left(\frac{3k_4^2+2k_4(k_1+k_2+k_3)+2(k_1k_2+k_2k_3+k_3k_1)}{2k_4^2}\right)\\
k_1^2+k_2^2+k_3^2+2k_4(k_1+k_2+k_3)+7k_4^2 &= 6k_4^2+4k_4(k_1+k_2+k_3)+2(k_1k_2+k_2k_3+k_3k_1)\\
k_1^2+k_2^2+k_3^2+k_4^2 &= (k_1+k_2+k_3+k_4)^2-(k_1^2+k_2^2+k_3^2+k_4^2)\\
2(k_1^2+k_2^2+k_3^2+k_4^2) &= (k_1+k_2+k_3+k_4)^2
\end{aligned}$$

□

13 三角形邊角關係

13.1 内外心與邊角距離關係

13.2 三角形點邊連線長關係

14 蝴蝶定理

15 帕斯卡定理

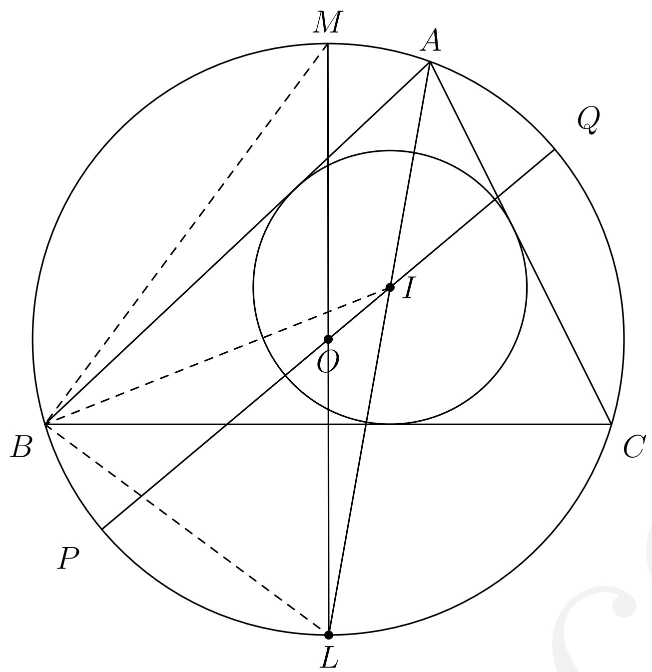
16 歐拉定理

引理 16.1 (雞爪定理)**.**

定理 16.2 (Euler's Theorem)**.** 對於任意三角形，

$$\overline{IO}^2 = R(R-2r)$$

其中，I 爲內心、O 爲外心、R 爲外接圓半徑、r 則是內切圓半徑



Proof.

(1) 做 $\overrightarrow{AI}$ 交圓 O 於 L，且 L 爲 BC 弧之中點

(2) 延長 $\overrightarrow{LO}$，交圓 O 於 M

(3) 過 I 做 $\overline{AB}$ 之垂足 H

$\angle IAH = \angle LMB$, $\angle AHI = \angle MBL$

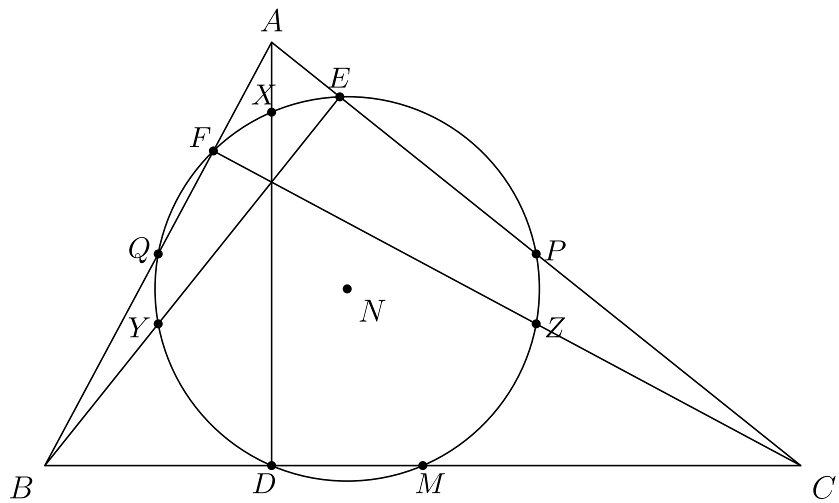
$\Rightarrow \triangle AHI \sim \triangle MBL\,(AA)$

$$\frac{\overline{IH}}{\overline{BL}} = \frac{\overline{AI}}{\overline{ML}} \quad \Rightarrow \quad \overline{AI} \times \overline{BL} = \overline{IH} \times \overline{ML} = r \times 2R \qquad (1)$$

□

17 歐拉線與九點圓

定理 17.1 (Euler Line)**.** 外心(O)、重心(G)、垂心(H)三點共線，且$2\overline{OG} = \overline{GH}$



18 卡諾定理

引理 18.1. $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

Proof.

$$\begin{aligned}
\sin A + \sin B + \sin C &= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} + \sin(\pi - (A+B)) \\
&= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} + 2\sin\frac{A+B}{2}\cos\frac{A+B}{2} \\
&= 2\sin\frac{A+B}{2}\left(\cos\frac{A-B}{2} + \cos\frac{A+B}{2}\right) \\
&= 2\sin\frac{\pi - C}{2} \times 2\cos\frac{A}{2}\cos\frac{B}{2} \\
&= 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}
\end{aligned}$$

□

引理 18.2. 對於任意三角形，內切圓半徑 r 與外接圓半徑 R 有以下關係：

$$r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

Proof.

$\triangle = \frac{1}{2}r(a+b+c) = \frac{1}{2}ab\sin C$

由正弦定理 $\Rightarrow a = 2R\sin A,\ b = 2R\sin B,\ c = 2R\sin C$

$2Rr(\sin A + \sin B + \sin C) = 2R\sin A \times 2R\sin B \times 2R\sin C$

$4r\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = 16R\sin\frac{A}{2}\cos\frac{A}{2}\sin\frac{C}{2}\cos\frac{C}{2}\sin\frac{C}{2}\cos\frac{C}{2}$

$r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

□

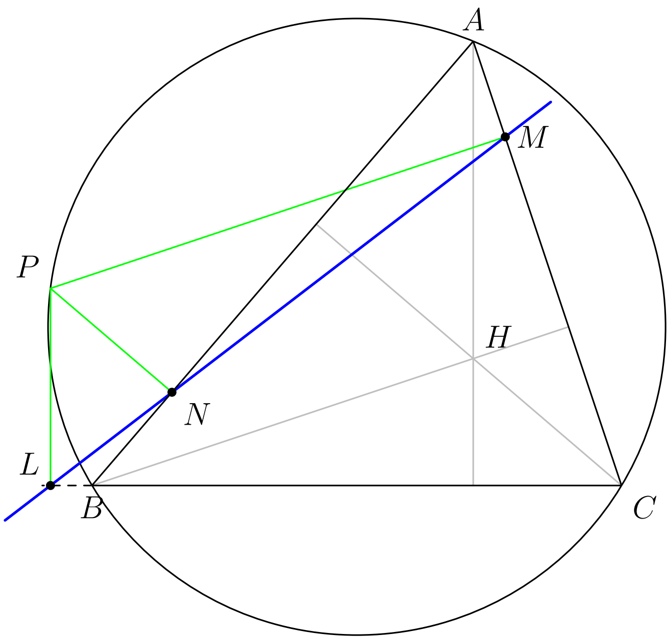
定理 18.3 (Carnot Theorem)**.** 三角形之外心到三邊的距離和爲 $R + r$

19 費馬點

20 垂足三角形

21 西姆松定理

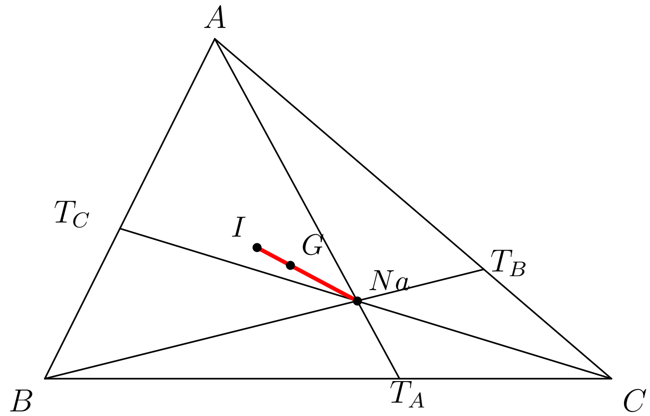
定理 21.1 (Simson's Theorem)**.** 平面上有一三角形 $\triangle ABC$ 以及一點 P，則 P 對 $\triangle ABC$ 之三邊做垂足共線若且唯若 P 在 $\triangle ABC$ 之外接圓上



22 拿破崙定理

23 旁切圓與中界線

24 奈格爾點 (Nagel Point)



定理 24.1. 內心 (I)、重心 (G)、奈格爾點 (Na) 共線，且 $\overline{NaG} = 2\,\overline{IG}$

Proof. 假設斜座標 $S \equiv \left\{B; \overrightarrow{BC}, \overrightarrow{BA}\right\}$

$$\overline{AT_B} = s - c,\ \overline{T_BC} = s - a,\ \overline{BT_A} = s - c$$

$$\Rightarrow \frac{s-c}{s-a} \times \frac{a}{s-c} \times \frac{\overline{T_ANa}}{\overline{NaA}} = 1 \quad (\text{孟氏定理})$$

$$\Rightarrow \frac{\overline{ANa}}{\overline{NaT_A}} = \frac{a}{s-a}$$

$$\text{又}\ \overrightarrow{BT_A} = \frac{s-c}{a}\overrightarrow{BC}$$

$$\begin{aligned}\Rightarrow \overrightarrow{BNa} &= \frac{a}{a+(s-a)}\overrightarrow{BT_A} + \frac{s-a}{a+(s-a)}\overrightarrow{BA} \\ &= \frac{a}{s} \times \frac{s-c}{a}\overrightarrow{BC} + \frac{s-a}{s}\overrightarrow{BA} \\ &= \frac{s-c}{s}\overrightarrow{BC} + \frac{s-a}{s}\overrightarrow{BA}\end{aligned}$$

$$\begin{aligned}\text{又}\ \overrightarrow{BI} &= \frac{c}{a+b+c}\overrightarrow{BC} + \frac{a}{a+b+c}\overrightarrow{BA} \\ &= \frac{c}{2s}\overrightarrow{BC} + \frac{a}{2s}\overrightarrow{BA}\end{aligned}$$

$$\text{又}\ \overrightarrow{BG} = \frac{1}{3}\overrightarrow{BC} + \frac{1}{3}\overrightarrow{BA}$$

$$\begin{aligned}\triangle GINa &= \left|\frac{1}{2}\begin{vmatrix} \frac{1}{3} & \frac{c}{2s} & \frac{s-c}{s} & \frac{1}{3} \\ \frac{1}{3} & \frac{a}{2s} & \frac{s-a}{s} & \frac{1}{3}\end{vmatrix}\right| \\ &= \left|\frac{1}{2}\left[\frac{a}{6s} + \frac{c(s-a)}{2s^2} + \frac{s-c}{3s} - \frac{c}{6s} - \frac{a(s-c)}{2s^2} - \frac{s-a}{3s}\right]\right| \\ &= \left|\frac{1}{12s^2}\left[sa + 3cs - 3ca + 2s^2 - 2sc - sc - 3as + 3ac - 2s^2 + 2as\right]\right| \\ &= 0\end{aligned}$$

$$\triangle GINa = 0 \quad \text{若且唯若} \quad G, I, Na\,\text{三點共線} \tag{1}$$

$$\begin{aligned}
\overline{NaG}^2 &= \left(\frac{s-c}{s} - \frac{1}{3}\right)^2 + \left(\frac{s-a}{s} - \frac{1}{3}\right)^2 \\
&= \left[\frac{3(s-c)-s}{3s}\right]^2 + \left[\frac{3(s-a)-s}{3s}\right]^2 \\
&= \frac{(2s-3c)^2 + (2s-3a)^2}{9s^2} := \frac{1}{9}\,\mathcal{K} \\
\overline{NaG}^2 &= \left(\frac{c}{2s} - \frac{1}{3}\right)^2 + \left(\frac{a}{2s} - \frac{1}{3}\right)^2 \\
&= \left[\frac{3c-2s}{6s}\right]^2 + \left[\frac{3a-2s}{6s}\right]^2 \\
&= \frac{(2s-3c)^2 + (2s-3a)^2}{36s^2} := \frac{1}{36}\,\mathcal{K} \\
&\Rightarrow \frac{\overline{NaG}^2}{\overline{GI}^2} = \frac{\frac{1}{9}\mathcal{K}}{\frac{1}{36}\mathcal{K}} = 4 \\
&\Rightarrow \sqrt{\overline{NaG}^2} = \sqrt{4\,\overline{GI}^2}
\end{aligned}$$

$$\overline{NaG} = 2\,\overline{GI} \tag{2}$$

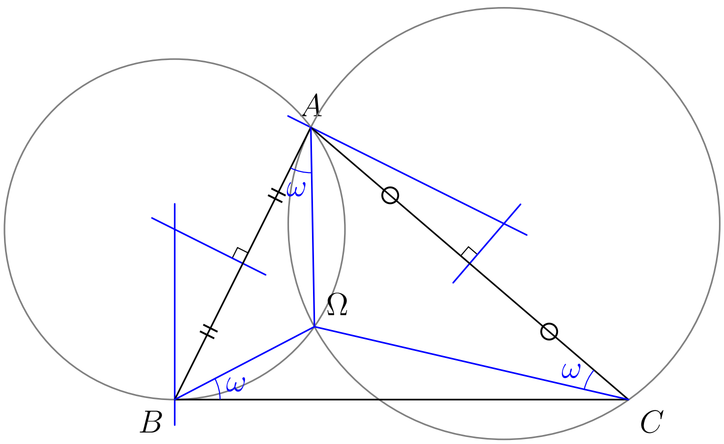
由(1)及(2)可知，G, I, Na 三點共線，且 $\overline{NaG} = 2\,\overline{IG}$

□

25 布洛卡兒點 (Brocard Point)

定義. 若一點到三頂點的線段與邊夾角相同，稱此點爲布洛卡兒點*(Brocard Point)*

作圖.



定理 25.1. $\cot\omega = \cot A + \cot B + \cot C$

Proof. $\angle AB\Omega = B - \omega \Rightarrow \angle A\Omega B = \pi - (B-\omega) - \omega = \pi - B$

$$\begin{cases} \dfrac{\overline{A\Omega}}{\sin(B-\omega)} = \dfrac{\overline{B\Omega}}{\sin\omega} \\ \dfrac{\overline{B\Omega}}{\sin(C-\omega)} = \dfrac{\overline{C\Omega}}{\sin\omega} \\ \dfrac{\overline{C\Omega}}{\sin(A-\omega)} = \dfrac{\overline{A\Omega}}{\sin\omega} \end{cases} \Rightarrow \begin{cases} \dfrac{\overline{A\Omega}}{\overline{B\Omega}} = \dfrac{\sin(B-\omega)}{\sin\omega} \\ \dfrac{\overline{B\Omega}}{\overline{C\Omega}} = \dfrac{\sin(C-\omega)}{\sin\omega} \\ \dfrac{\overline{C\Omega}}{\overline{A\Omega}} = \dfrac{\sin(A-\omega)}{\sin\omega} \end{cases}$$

$$\Rightarrow \sin(A-\omega)\,\sin(B-\omega)\,\sin(C-\omega) = \sin^3\omega$$

$$\begin{cases} \dfrac{\overline{C\Omega}}{\sin(A-\omega)} = \dfrac{\overline{AC}}{\sin(\pi-A)} = \dfrac{\overline{AC}}{\sin A} \\ \dfrac{\overline{C\Omega}}{\sin\omega} = \dfrac{\overline{BC}}{\sin(\pi-C)} = \dfrac{\overline{BC}}{\sin C} \end{cases}$$

$$\Rightarrow \frac{\sin(A-\omega)}{\sin\omega} = \frac{\overline{BC}\,\sin A}{\overline{AC}\,\sin C} = \frac{2R\sin A\cdot\sin A}{2R\sin B\cdot\sin C}$$

$$\Rightarrow \frac{\sin(A-\omega)}{\sin\omega} = \frac{\sin^2 A}{\sin B\cdot\sin C} = \frac{\sin A\cdot\sin\left(\pi-(B+C)\right)}{\sin B\cdot\sin C}$$

$$\Rightarrow \frac{\sin A\cos\omega - \cos A\sin\omega}{\sin\omega} = \frac{\sin A\cdot(\sin B\cos C + \cos B\sin C)}{\sin B\cdot\sin C}$$

$$\Rightarrow \sin A\cot\omega - \cos A = \sin A\cdot\sin\left(\cot C + \cot B\right)$$

$$\Rightarrow \cot\omega - \cot A = \cot C + \cot B$$

$$\Rightarrow \cot\omega = \cot A + \cot B + \cot C$$

□