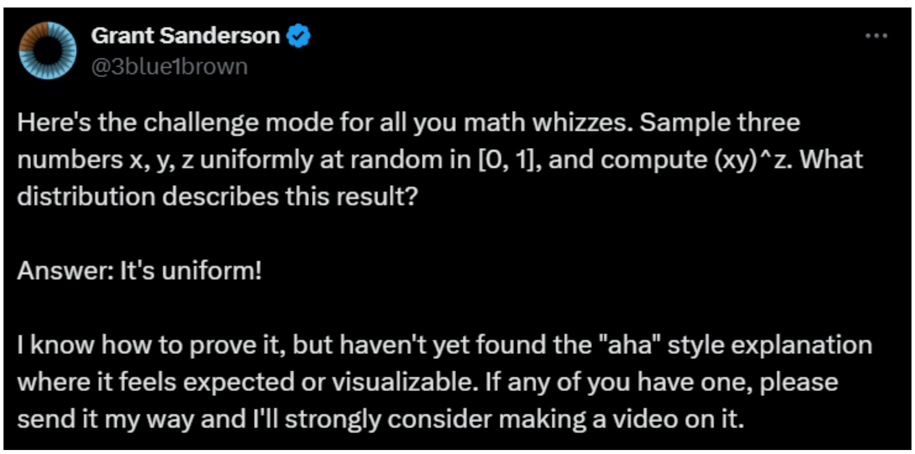
Magical Distribution

Chocomint

1 Inspiration

The problem is given by Grant Sanderson (3B1B):


Grant Sanderson
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Here's the challenge mode for all you math whizzes. Sample three numbers x, y, z uniformly at random in [0, 1], and compute (xy)^z. What distribution describes this result?

Answer: It's uniform!

I know how to prove it, but haven't yet found the "aha" style explanation where it feels expected or visualizable. If any of you have one, please send it my way and I'll strongly consider making a video on it.

2 CDF and PDF

Given a random variable X with *probability desity function (PDF)* $f_X(x)$ and *cumulative distribution function (CDF)* $F_X(x)$, they have following relationship:

$$f_X(x) = \frac{d}{dx} F_X(x)$$

3 Calculation

We have 3 random variable $x, y, z \sim U(0,1)$, which can be regard as "choosing random point in the unit cube $\{(x,y,z)\,|\,0 \le x,y,z \le 1\}$."

Now, let's consider our goal function $(xy)^z$. We can derive its CDF by

$$\text{CDF of } (xy)^z = F(s) = \mathrm{P}\big[(xy)^z \le s\big] \tag{1}$$

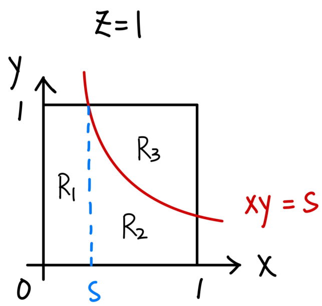
Notice that $\mathrm{P}\big[(xy)^z \le s\big]$ is equal to the volume ratio of $\mathrm{V}\big[(xy)^z \le s \text{ within unit cube}\big]$ to V(unit cube). Because V(unit cube) = 1, we get

$$\mathrm{P}\big[(xy)^z \le s\big] = \mathrm{V}\big[(xy)^z \le s \text{ within unit cube}\big] \equiv V_s$$

Consider the inequality $(xy)^z \le s$. Apply ln() and divide $\ln(xy)$ to both side to leave z to one side. Be careful that $x, y, s \le 1$ which means dividing $\ln(xy)$ will change the sign. Therefore, we get the final inequality to represent our volume:

$$V_s = \left\{ z \ge \frac{\ln s}{\ln xy} \,\middle|\, \text{within unit cube} \right\} \tag{2}$$

Using this inequality, we can easily express the volume by double integral. But cause z also has limitation $(z \le 1)$, we should divide our integral domain.



As the picture shows, red line is the intersection line of $(xy)^z$ on the plane $z = 1$ and blue dotted line is $x = s$. Using these two lines, we separate the integral domain into three parts R_1, R_2, R_3 (project to x-y plane). Therefore, the volume is

$$V_s = 1 - \left(\iint_{R_1} \frac{\ln s}{\ln xy}\, dA + \iint_{R_2} \frac{\ln s}{\ln xy}\, dA + \iint_{R_3} 1 \cdot dA \right) \tag{3}$$

Expand the integrals by implementing upper and lower limits:

$$1 - V_s = \int_0^s \int_0^1 \frac{\ln s}{\ln xy}\, dy\, dx + \int_s^1 \int_0^{s/x} \frac{\ln s}{\ln xy}\, dy\, dx + \text{Area of } R_3$$

Trivially, the area of R_3 is $(s \ln s - s + 1)$. Therefore, we can simplify the equation:

$$-V_s = (\ln s) \left(\int_0^s \int_0^1 \frac{1}{\ln xy}\, dy\, dx + \int_s^1 \int_0^{s/x} \frac{1}{\ln xy}\, dy\, dx \right) + (s \ln s - s) \tag{4}$$

Extract the two integrals and denote them as I_1 and I_2:

$$I_1 = \int_0^s \int_0^1 \frac{1}{\ln xy}\, dy\, dx \qquad I_2 = \int_s^1 \int_0^{s/x} \frac{1}{\ln xy}\, dy\, dx \tag{5}$$

First, we consider the derivative of I_1.

$$\frac{dI_1}{ds} = \frac{d}{ds} \int_0^s \int_0^1 \frac{1}{\ln xy}\, dy\, dx = \int_0^1 \frac{1}{\ln sy}\, dy$$

Because inner integral is independent to s, we can just simply apply FTOC to replace x with s in inner integral.

Next, we move to the derivative of I_2.

$$\frac{dI_2}{ds} = \frac{d}{ds} \int_s^1 \int_0^{s/x} \frac{1}{\ln xy}\, dy\, dx$$

Unfortunately, there's s in inner integral, so we cannot apply FTOC. Let's introduce a useful tool, Leibniz integral rule (simplified form):

$$\boxed{\frac{d}{dx} \int_a^x f(x,t)\, dt = f(x,x) + \int_a^x \frac{\partial}{\partial x} f(x,t)\, dt} \tag{6}$$

With this tool, we represent inner integral as $f(s, x)$, then we can get

$$\begin{aligned}
\frac{dI_2}{ds} &= -\frac{d}{ds} \int_1^s f(s,x)\, dx \\
&= -f(s,s) - \int_1^s \frac{\partial}{\partial s} f(s,x)\, dx \\
&= -\int_0^{s/s} \frac{1}{\ln sy}\, dy - \int_1^s \frac{1}{x} \cdot \frac{1}{\ln s}\, dx \\
&= -\int_0^1 \frac{1}{\ln sy}\, dy - 1
\end{aligned}$$

From line 2 to 3, we apply FTOC again. Let's combine the derivative of I_1 and I_2:

$$\frac{dI_1}{ds} + \frac{dI_2}{ds} = \frac{d}{ds}(I_1 + I_2) = -1 \tag{7}$$

That is a constant! On the other hand, it shows that

$$I_1 + I_2 = -s + Const.$$

Notice that I_1 has the integral $\int_0^s$ and I_2 has $\int_0^{s/x}$, so when $s = 0$, I_1 and I_2 should be both 0. That is to say, $Const. = 0$, which also means

$$I_1 + I_2 = -s \tag{8}$$

Back to the equation (4), we can now derivative both side by s:

$$-\frac{dV_s}{ds} = \frac{1}{s}(I_1 + I_2) + (\ln s)\frac{d}{ds}(I_1 + I_2) + \ln s$$

Insert the values that we just calculated, then we get

$$\frac{dV_s}{ds} = 1$$

Recall the definition of CDF, which is just the volume V_s, and PDF is the derivative of CDF. Therefore:

$$\text{PDF of } (xy)^z = 1 \tag{9}$$

That is to say, the distribution of $(xy)^z$ is $U(0, 1)$